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LETTER TO THE EDITOR

On the integrable equations and degenerate dispersion laws in multidimensional spaces

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Abstract. The general reason for the restricted applicability of the inverse scattering transform method in multidimensional spaces is discussed.

The inverse scattering transform (IST) method is a powerful tool for the investigation of nonlinear evolution equations (see e.g. Zakharov *et al* 1980, Bullough and Caudrey 1980, Ablowitz and Segur 1981, Calogero and Degasperis 1982). The IST method has been applied to numerous nonlinear equations in $1+1$ and $1+2$ dimensions (one time and one and two spatial dimensions). However, the applicability of the IST method to the equations in more than two spatial dimensions is much more restricted (Zakharov *et al* 1980). For example, only linear differential equations can be represented in the Lax form $\partial L/\partial t=[L, A]$ if L is the multidimensional Schrödinger operator $L=-\Delta+U(x_1, \ldots, x_N)$ (Perelomov 1976).

In the present paper the general reason which leads to the strong restriction on the applicability of the standard version of the IST method in multidimensional spaces is pointed out. The structure of the degenerate dispersion laws in the case of more that two spatial dimensions is also considered.

In the standard version of the IST method the nonlinear equations are equivalent to the commutativity condition $[T_1, T_2]=0$ of the two operators T_1 and T_2. Here we will consider for definiteness the non-stationary Schrödinger operator

$$T_1=\partial/\partial x_N+\partial^2/\partial x_1^2+\cdots+\partial^2/\partial x_{N-1}^2+U(x_1, \ldots, x_N, t) \tag{1}$$

as the operator T_1. Let the operator T_2 be of the form

$$T_2=\partial/\partial t+\mathscr{F}(U(x_1, \ldots, x_N, t), \partial/\partial x_1, \ldots, \partial/\partial x_N). \tag{2}$$

The dimension N of the operator T_1 is an arbitrary one. For $N=2$ the operator (1) has been used for the integration of the Kadomtsev–Petviashvili equation (see Zakharov *et al* 1980). The direct and inverse scattering problems for the operator (1) at $N=2$ were considered by Zakharov and Manakov (1979) and Manakov (1981).

Let us introduce analogously to the case $N=2$ the solutions $F_{\boldsymbol{k}}^{\pm}(\boldsymbol{x}, x_N, t)$ of the scattering problem

$$T_1F(\boldsymbol{x}, x_N, t)=0 \tag{3}$$

given by their asymptotic behaviour

$$F_{\boldsymbol{k}}^{\pm}(\boldsymbol{x}, x_N, t)\xrightarrow[x_N\to\pm\infty]{}\exp(\mathrm{i}\boldsymbol{k}\boldsymbol{x}+\boldsymbol{k}^2x_N)$$

where $x=(x_1,\ldots,x_{N-1})$, $k=(k_1,\ldots,k_{N-1})$ and $-\infty<k_i<\infty$. The scattering matrix $S(k',k,t)$ is defined as follows:

$$F_k^+(x,x_N,t)=\int dk'\,F_{k'}^-(x,x_N,t)S(k',k,t). \tag{4}$$

For small (in a suitable sense) potentials one has in linear (Born) approximation the well known expression

$$S(k',k,t)=\delta(k'-k)+\tilde{U}(k'-k,k'^2-k^2,t) \tag{5}$$

where $\tilde{U}(q,q_N,t)$ is a Fourier transform of the potential $U(x,x_N,t)$.

The evolution of the scattering matrix in time t is defined by the operator T_2 and it is of the form

$$dS(k',k,t)/dt=(A(k')-A(k))S(k',k,t) \tag{6}$$

where $A(k)=\mathscr{F}(0,ik_1,\ldots,ik_{N-1},ik^2)$. The evolution law (6) is valid, of course, in the Born approximation too. Substituting (5) into (6), one gets

$$\partial\tilde{U}(q,q_N,t)/\partial t=(A(k')-A(k))\tilde{U}(q,q_N,t) \tag{7}$$

where $q_1=k_1'-k_1,\ldots,q_{N-1}=k_{N-1}'-k_{N-1}$, $q_N=k'^2-k^2$. For self-consistency of equation (7) it is necessary that $A(k')-A(k)$ be also a function only of the variables $q_1,\ldots,q_{N-1},q_N$, i.e.

$$A(k')-A(k)=W(k'-k,k'^2-k^2) \tag{8}$$

where $W(q,q_N)$ is a certain function.

So the fulfilment of (8) is the necessary condition for the self-consistency of the evolution law (6) of the scattering matrix. In other words, all admissible functions $A(k)$ and therefore all admissible operators T_2 should satisfy the condition (8).

The condition (8) is a functional equation for $A(k)$. We consider subsequently the cases $N=2$ and $N\geqslant 3$.

It is easy to see that in the two-dimensional case $N=2$ the condition (8) is always satisfied. Indeed, from the definitions $q_1=k_1'-k_1$, $q_2=k_1'^2-k_1^2$ one has $k_1'=(q_2+q_1^2)/2q_1$, $k_1=(q_2-q_1^2)/2q_1$. The quantity

$$A(k_1')-A(k_1)=A((q_2+q_1^2)/2q_1)-A((q_2-q_1^2)/2q_1)$$

is a function of only q_1 and q_2 for any function $A(k_1)$.

For multidimensional spaces ($N\geqslant 3$) the situation is quite different.

Theorem 1. For $N\geqslant 3$ the condition

$$A(k')-A(k)=W(k'-k,f(k')-f(k)) \tag{9}$$

where $f(k)$ is an arbitrary entire function is satisfied only for linear functions $A(k)=\sum_{i=1}^{N-1}\alpha_i k_i$ within the class of entire functions.

To prove this theorem let us consider firstly the case $N=3$ and $f(k_1,k_2)=k_1^2+k_2^2$. Let us introduce the variable $\tilde{q}_1=k_1'+k_1$ in addition to the variables $q_1=k_1'-k_1$, $q_2=k_2'-k_2$, $q_3=k_1'^2+k_2'^2-k_1^2-k_2^2$. Expressing q_1, q_2, q_3, $\tilde{q}_1$ through q_1, q_2, q_3, $\tilde{q}_1$

one has

$$k'_1 = \tfrac{1}{2}\tilde{q}_1 + \tfrac{1}{2}q_1, \qquad k_1 = \tfrac{1}{2}\tilde{q}_1 - \tfrac{1}{2}q_1,$$
$$k'_2 = (q_2 + q_2^2 - \tilde{q}_1 q_1)/2q_2, \qquad k_2 = (q_3 - q_2^2 - \tilde{q}_1 q_1)/2q_2. \tag{10}$$

The sets of variables k'_1, k'_2, k_1, k_2 and q_1, q_2, q_3, $\tilde{q}_1$ are connected by a nondegenerate transformation and give different parametrisations of the same four-dimensional space.

The condition (9) is now the condition of independence of the quantity

$$A(\tfrac{1}{2}\tilde{q}_1 + \tfrac{1}{2}q_1, (q_3 + q_2^2 - \tilde{q}_1 q_1)/2q_2) - A(\tfrac{1}{2}\tilde{q}_1 - \tfrac{1}{2}q_1, (q_3 - q_2^2 - \tilde{q}_1 q_1)/2q_2) \tag{11}$$

on the variable $\tilde{q}_1$. For a linear function $A = \alpha_1 k_1 + \alpha_2 k_2$ the quantity (11) is $\alpha_1 q_1 + \alpha_2 q_2$, i.e. it does not depend on $\tilde{q}_1$. For a quadratic function $A = \alpha k_1^2 + \beta k_1 k_2 + \gamma k_2^2$ the quantity (11) contains the following dependence on $\tilde{q}_1$:

$$\alpha q_1 \tilde{q}_1 + \tfrac{1}{2}(\beta - \gamma) q_2 \tilde{q}_1 - \tfrac{1}{2}\beta (q_1^2/q_2)\tilde{q}_1. \tag{12}$$

The quantity (12) should be equal to zero for any q_1, q_2. Therefore, $\alpha = \beta = \gamma = 0$.

For any polynomial function A the situation is similar. Indeed, let $A(\boldsymbol{k})$ be a polynomial of order n. Let us introduce the variable $\tilde{q}_1 = k'_1 + k_1$ in addition to the variables q_1, q_2, q_3. The condition of the independence of the quantity (11) of $\tilde{q}_1$ is equivalent to the equations

$$(\partial^m/\partial \tilde{q}_1^m)(A(k'_1, k'_2) - A(k_1, k_2))|_{\tilde{q}_1 = 0} = 0 \tag{13}$$

where k'_1, k'_2, k_1, k_2 are given by (10). The left-hand side of (13) is of the form $\Sigma_{m_1+m_2+m_3=m} C_{m_1 m_2 m_3} q_1^{m_1} q_2^{m_2} q_3^{m_3}$. Since all powers of q_1, q_2, q_3 are independent, (13) for given m is in fact a system of $\frac{1}{2}(m+1)(m+2)$ equations. So for $n > 1$ a number of the equations for the coefficients of the polynomial $A(k_1, k_2)$ is more than the number $(n+1)$ of these coefficients. Therefore (13) has only trivial (linear) solutions for $n > 1$.

The proof of the theorem for $N > 3$ and arbitrary function $f(\boldsymbol{k})$ is quite similar. We introduce the new variables $\tilde{q}_1 = k'_1 + k_1, \ldots, \tilde{q}_{N-2} = k'_{N-2} + k_{N-2}$ in addition to the variables $q_1, \ldots, q_{N-1}, q_N$. The sets of the variables $k'_1, \ldots, k'_{N-1}, k_1, \ldots, k_{N-1}$ and $q_1, \ldots, q_N, \tilde{q}_1, \ldots, \tilde{q}_{N-2}$ are connected by a non-degenerate transformation and give different parametrisations of the same $(2N-2)$-dimensional space. The condition (9) is the condition of the independence of $A(\boldsymbol{k}') - A(\boldsymbol{k})$ (where $\boldsymbol{k}'$ and $\boldsymbol{k}$ are expressed through $q_1, \ldots, q_N, \tilde{q}_1, \ldots, \tilde{q}_{N-2}$) of the variables $\tilde{q}_1, \ldots, \tilde{q}_{N-2}$, that is equivalent to the equations

$$(\partial^m/\partial \tilde{q}_l^m)(A(\boldsymbol{k}') - A(\boldsymbol{k}))|_{\tilde{q}_l = 0} = 0, \qquad l = 1, \ldots, N-2. \tag{14}$$

It is not difficult to show that for polynomial $A(\boldsymbol{k})$ of second and higher orders the number of equations (14) for the coefficients of this polynomial is much greater than the number of these coefficients and equations (14) are satisfied only when all these coefficients are equal to zero. Therefore equations (14) have only the trivial solution $A = \Sigma\, \alpha_i k_i$.

Thus we see that there exist strong restrictions on the form of the function $A(\boldsymbol{k})$ in multidimensional spaces ($N \geq 3$). As a result only the operators T_2 which are linear on $\partial/\partial x_1, \ldots, \partial/\partial x_{N-1}$ are admissible for $N \geq 3$. It is easy to show then that only trivial linear equations $\partial u/\partial t + \Sigma_{i=1}^{N-1} \alpha_i \partial u/\partial x_i = 0$ can be represented in the form $[T_1, T_2] = 0$ with the use of these admissible operators T_2.

So the nature of the restrictions on the applicability of the IST method in multidimensional spaces is clear already in the Born approximation.

Similar results are valid for other multidimensional scattering problems too. For the problem $\Sigma_{i=1}^{N} A_i \partial\psi/\partial x_i + P(x_1, \ldots, x_N, t)\psi = 0$ see Konopelchenko (1983).

Let us now discuss the properties of the dispersion laws. From formulae (7) and (9) it follows that $W(\boldsymbol{q}, q_N)$ is nothing but the dispersion law for the corresponding evolution equation. One can obtain from (9) the equation which contains only the function $W(\boldsymbol{q}, q_N)$. indeed, putting $\boldsymbol{k} = 0$ in (9) one gets

$$A(\boldsymbol{k}') - A(0) = W(\boldsymbol{k}', f(\boldsymbol{k}')). \tag{15}$$

For $\boldsymbol{k}' = 0$ from (9) we have

$$A(0) - A(\boldsymbol{k}) = W(-\boldsymbol{k}, -f(\boldsymbol{k})). \tag{16}$$

Substituting (15) and (16) into (9) we obtain

$$W(\boldsymbol{k}', f(\boldsymbol{k}')) + W(-\boldsymbol{k}, -f(\boldsymbol{k})) = W(\boldsymbol{k}' - \boldsymbol{k}, f(\boldsymbol{k}') - f(\boldsymbol{k})). \tag{17}$$

Denoting $p' \stackrel{\text{def}}{=} (p'_1, \ldots, p'_{N-1}, p'_N) = (k'_1, \ldots, k'_{N-1}, f(\boldsymbol{k}'))$ and $p \stackrel{\text{def}}{=} (p_1, \ldots, p_{N-1}, p_N) = (-k_1, \ldots, -k_{N-1}, -f(\boldsymbol{k}))$ we rewrite (17) as

$$W(p + p') = W(p) + W(p') \tag{18}$$

i.e. as the decay equation. The dispersion laws with the properties (18) have been discussed recently in Zakharov and Schulman (1980) and Zakharov (1982). They are interested in the degenerate dispersion laws, i.e. the dispersion laws for which equation (18) has several (not the only) solutions.

It is easy to see that use of equation (9) gives us the solution of (18). Indeed, let us introduce the variables $\boldsymbol{k}'$, $\boldsymbol{k}''$, $\boldsymbol{k}$ such that

$$\boldsymbol{p} = \boldsymbol{k}'' - \boldsymbol{k}, \qquad \boldsymbol{p}' = \boldsymbol{k}' - \boldsymbol{k}'', \qquad p_{N-1} = f(\boldsymbol{k}'') - f(\boldsymbol{k}), \qquad p'_{N-1} = f(\boldsymbol{k}') - f(\boldsymbol{k}'') \tag{19}$$

where $f(\boldsymbol{k})$ is a certain function of $\boldsymbol{k}$. Let $A(\boldsymbol{k})$ be a solution of (18). Then

$$W(\boldsymbol{p}, p_N) = A(\boldsymbol{k}'') - A(\boldsymbol{k}) \tag{20}$$

is the solution of (18).

So any solution of the problem (9) gives us the solution of equation (18). In particular, if several functions A exist which satisfy (9) then the dispersion law $W(p)$ is degenerate (with respect to the decay process (18)). We see that the problem of enumeration of the evolution equations integrable by a given spectral problem is closely connected to the problem of degenerate dispersion laws.

In two-dimensional space ($N = 2$) equation (9) has an infinite number of solutions and therefore the formulae (19)–(20) give the family of degenerate dispersion laws (see Zakharov and Schulman 1980 and Zakharov 1982).

For multidimensional spaces ($N \geq 3$), from theorem 1 one obviously has the following.

Theorem 2. In multidimensional spaces ($N \geq 3$) there exist no degenerate dispersion laws of the form (19)–(20).

This theorem has a corollary which was discussed previously by Zakharov (1982).

Corollary. For $N=3$ there exist no degenerate dispersion laws of the form

$$W(q_1, q_3, \eta) = A(k_1') - A(k_1) + \alpha\eta + \sum_{n=1}^{\infty} W_n(k_1', k_1)\eta^{n+1}, \tag{21}$$

$$q_1 = k', -k_1, \qquad q_3 = f(k_1') - f(k_1),$$

where $W_n \neq 0$.

To prove this corollary let us expand the functions W and A given by (19) and (20) for $N=3$ in the power series of $q_2 = k_2' - k_2 = \eta$. One has

$$\begin{aligned} W(q_1, q_2, q_3) &= W_0(q_1, q_3) + \eta \sum_{n=0}^{\infty} W_{(n)}(q_1, q_3)\eta^n, \\ A(k_1', k_2') &= A(k_1', k_2) + \eta \sum_{n=0}^{\infty} A_{(n)}(k_1', k_2)\eta^n. \end{aligned} \tag{22}$$

Substituting (22) into (20) we get

$$W_0(q_1, q_3) + \eta \sum_{n=0}^{\infty} W_{(n)}(q_1, q_3)\eta^n = A(k_1', k_2) - A(k_1, k_2) + \eta \sum_{n=0}^{\infty} A_{(n)}(k_1', k_2)\eta^n. \tag{23}$$

At zero order of η one has

$$\begin{aligned} &W_0(q_1, q_3) = A(k_1') - A(k_1), \\ &q_1 = k_1' - k_1, \qquad q_3 = f(k_1') - f(k_1). \end{aligned} \tag{24}$$

Then since $A(k_1, k_2)$ should be a linear function of k_1, k_2, i.e. $A_{(1)} = A_{(2)} = \cdots = 0$, we obtain $W_{(1)} = W_{(2)} = \cdots = 0$. That is the statement of Zakharov's theorem 2.2 (Zakharov 1982).

In the two-dimensional case, $N=2$, there exist degenerate dispersion laws of the form $W(q_1, q_2) = q_1 F(q_2/q_1)$ (Zakharov 1982). In multidimensional spaces ($N \geqslant 3$) similar degenerate dispersion laws do not exist. Indeed, letting $q_i = k_i' - k_i \to 0$ ($i = 1, \ldots, N-1$), one has

$$q_N = \sum_{i=1}^{N-1} \frac{\partial f(\boldsymbol{k})}{\partial k_i} q_i, \tag{25}$$

$$W(q_1, \ldots, q_N) = \sum_{i=1}^{N-1} \frac{\partial A(\boldsymbol{k})}{\partial k_i} q_i. \tag{26}$$

This system of equations is under determined. It is impossible to express $k_1, \ldots, k_{N-1}$ through $q_1, \ldots, q_N$ with only the use of equation (25). As a result the right-hand side of (26) is a function of $q_1, \ldots, q_N$ only for the linear function $A(k_1, \ldots, k_{N-1})$.

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